STELLAR STRUCTURE

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1 INTRODUCTION

The basics are covered and we are all ready to not only travel to the far stars but dig up their surface to study the real magic. Here, you should stop and contemplate the immense contribution by scientists to understand the processes in the most indirect ways possible. We will deal with stellar structures and processes happening throughout the lifetime of a star and why on the first place it happens.

2 TOPICS TO STUDY

1. Hydrostatic equilibrium, Timescales: dynamical, thermal, nuclear

2. Energy generation, thermonuclear reactions

3. Energy transport; opacity, radiative and convective transport

4. Equations of stellar structure

5. Virial Theorem, Pressure

6. Stellar properties as a function of mass, homology

6. Degeneracy: Chandrasekhar limit

3 Hydrostatic Equilibrium

We are discussing the hydrostatic equilibrium first. Stars are composed of plasma which being a fourth state is considered as a fluid. This whole struc-

ture is held together by gravity which is not opposed by one but three different types of pressure.
Three types of stellar pressure are :

1. Ideal Pressure - Collision b/w gas particles ($P_{ideal} \propto \rho T$)

2. Radiation Pressure - Collision b/w photons and matter ($P_{rad} \propto \rho T^4$)

3. Degeneracy Pressure - Result of resistance of electrons or neutrons against being compressed to a smaller volume

($P_{deg} \propto \rho^{5/3}$ when non-relativistic)
($P_{deg} \propto \rho^{4/3}$ when relativistic)

A star's interior stability is promised by the balancing pressure and gravity. The star is stabilized(i.e.,nuclear reactions are kept under control) by a pressure-temperature thermostat or is self - regulating.

We shall discuss first about a thin shell of material in the sphere. Let the shell be at a distance r in a sphere of M and R having thickness dr. Mass of the shell $dM = 4\pi r^3 \rho dr$. The shell is subject to gravitational force and pressure both acting inwards.

$$F_g = -\frac{GM\rho}{r^2}\,dr \qquad F_p = -\frac{dP}{dr}\,dr \tag{1}$$

We already know $F_p = P(r) - P(r + dr)$ as pressure at r is more than at r+dr. Using F = ma we can calculate acceleration of the shell.

$$(\rho dr)\ddot{r} = -[\frac{GM\rho}{r^2} + \frac{dP}{dr}]dr$$

Therefore, acceleration is

$$\ddot{r} = -[\frac{GM}{r^2} + \frac{dP}{\rho dr}]$$

This is the equation of motion of the shell. But when calculating equation for hydrostatic equilibrium of the star we nullify the acceleration part and equate the pressure applied and the gravitational force.

$dP.A = -GM(r)m/r^2 = -GM(r) \times (\rho A dr)/r^2$

$dP/dr = -GM(r)\rho(r)/r^2$

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \tag{2}$$

Changing in terms of dm we will have Lagrangian form of the equation

$dP/dm = (dP/dr)(dr/dm) = (dP/dr)(dm/dr)^{-1}$

$-(GM\rho)/(r^2) \times (1/4\pi r^2\rho) = -(GM)/(4\pi r^4)$

The initial one was Euler form. Now, a simple trick and we get to know the central temperature of a star.
For starters, we assume the density to be uniform

$\rho = constant$
$dP/dr = -GM(r)\rho/r^2$
$M(r) = 4\pi r^3\rho/3$
$dP/dr = -G(4\pi r^3\rho^2)/3r^2$

$$\int_{P_c}^{0} dP = \int_{0}^{R} \frac{-4\pi G\rho^2}{3} r dr$$

$$P_c = \frac{4\pi G\rho^2}{6} R^2 = \frac{G\rho}{2R} \times \frac{4\pi\rho R^3}{3}$$

Therefore,

$$P_c = \frac{GM\rho}{2R}$$

As $P_c = P_{ideal}$

$$P_c = 1.69\rho N_a k T_c = \frac{GM\rho}{2R}$$

$$T_c = \frac{GM}{3.38 N_a k R}$$

Lagrangian form is better in case of stars, the mass parameter is the independent coordinate and others are a function of it. We label each mass shell by the mass m interior to it. Thus for a star of total mass M, the shell m = 0 is the one at the center of the star, the one m = M/2 is at the point that contains half the mass of the star, and the shell m = M is the outermost one.

4 The Virial Theorem and it's Implications

Hydrostatic equilibrium helps in linking gravitational potential energy and internal energy. The virial theorem provides an equation that clarifies the relationship between average over time of the total kinetic energy of distinct particles to that of the total potential of the system. We can also look in the alternative ways. Multiply the equation of hydrostatic equilibrium on both sides by V.

$$\int_0^P VdP = -\int_0^P \frac{GMV}{4\pi r^4}dm = -\int_0^P \frac{GM}{3r}dm \tag{3}$$

Here the gravitational potential will indicate the energy required to assemble the matter from infinity into a star.

$$\Omega = -\int_0^M \frac{GMdm}{r} \tag{4}$$

$$\int_0^{P(R)} VdP = [PV]_0^R - \int_0^{V(R)} PdV \tag{5}$$

as at boundary condition P = 0 and at centre V = 0 therefore the first term at right hand vanishes giving

$$\Omega = -3\int_0^{V(R)} PdV = -3\int_0^{V(R)} \frac{Pdm}{\rho} \tag{6}$$

This is the general form used. Using $P = \rho k_B T/\mu m_H = R\rho T/\mu$ which is applicable for ideal gas and substituting in the derived equation of virial theorem(22) we will get

$$\int_0^M \frac{RTdm}{\mu} = -\frac{\Omega}{3} \tag{7}$$

For a monatomic ideal gas, the internal energy per particle is $(3/2)k_bT$, so the internal energy per unit mass is $u = (3/2)RT/\mu$. We will get

$$\int_0^M \frac{2udm}{3} = -\frac{\Omega}{3} \qquad U = -\frac{\Omega}{2} \tag{8}$$

Therefore the total energy $E = U + \Omega$. Note that, since $\Omega < 0$, this implies that the total energy of a star made of ideal gas is negative, which makes sense given that a star is a gravitationally bound object. As we have only talked about whole radius condition we should also take one example where $R_s < R$. We will get

$$P_sV_s - \int_0^{V_s} \frac{P}{\rho} dm = \frac{\Omega_s}{3} \tag{9}$$

As can be concluded P_s and V_s are for pressure(exerted by enveloping sphere) and volume of shell s.

We know that $U = 3MR\overline{T}/2\mu$ and $\Omega = -\alpha GM^2/R$ which help us to know the average temperature of the star(solve the eqn).

$$\overline{T} = \frac{\alpha\mu GM}{3R^2} \tag{10}$$

The result $E = U + \Omega$ bears a significant resemblance to one that applies to orbits. In an orbit $K = mv^2/2 = GmM/2R$ and potential yielding to $-GmM/R$ which is the same. Therefore it hints at an alternative proof for a system of particles.

$$K = \frac{\sum_i^N m_i(\dot{x_i}^2 + \dot{y_i}^2 + \dot{z_i}^2)}{2} \qquad W = -(G/2)\sum_{i,j} \frac{m_i m_j}{|\vec{r_i} - \vec{r_j}|} \tag{11}$$

$$I = \sum_i^N m_i m_i(x_i^2 + y_i^2 + z_i^2) \qquad \frac{\mathrm{d}I}{\mathrm{d}t} = 2\sum_i^N m_i[x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i] \tag{12}$$

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2\sum_i^N m_i(\dot{x_i}^2 + \dot{y_i}^2 + \dot{z_i}^2 + x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i) \tag{13}$$

Accordingly, we will use acceleration for solving further.

$$\ddot{\vec{r_i}} = -G\sum_j \frac{m_j(\vec{r_i} - \vec{r_j})}{|\vec{r_i} - \vec{r_j}|^3} \tag{14}$$

Now , let us substitute this value in the following eqn :-

$$\sum m_i(x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i) = -G\sum_{i,j} \frac{m_i m_j}{|\vec{r}_{ij}|^3}[x_i(x_i - x_j) + y_i(y_i - y_j) + z_i(z_i - z_j)] \tag{15}$$

$$= -G\sum_{i,j}\frac{m_i m_j}{|\overrightarrow{r}_{ij}|^3}[x_i^2 + y_i^2 + z_i^2 - x_i x_j - y_i y_j - z_i z_j] \tag{16}$$

$$= -G\sum_{i,j}\frac{m_i m_j}{|\overrightarrow{r}_{ij}|^3}[x_j^2 + y_j^2 + z_j^2 - x_i x_j - y_i y_j - z_i z_j] \tag{17}$$

$$= -\frac{G}{2}\sum_{i,j}\frac{m_i m_j}{|\overrightarrow{r}_{ij}|^3}[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2] \tag{18}$$

Taking equation(17) (18), dividing by 2 and subtracting from the other we get perfect squares of r_i and r_j.

$$S = -\frac{G}{2}\sum_{i,j}\frac{m_i m_j}{|\overrightarrow{r}_{ij}|^3}(r_{ij})^2 = W \qquad \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + W \tag{19}$$

5 Energy generation and reactions

Hydrostatic balance is essentially a statement of conservation of momentum. We should discuss the first law of thermodynamics for proving the energy conservation which is universally true.

5.1 First law of Thermodynamics

$U = Q + PV$. We will consider aggain a thin spherical shell of mass dm. This mass element has an internal energy per unit mass u, so the total energy of the shell is udm. The internal energy can consist of thermal energy (i.e. heat) and chemical energy (i.e. the energy associated with changes in the chemical state of the gas). $\partial E = \partial(udm) = \partial udm$, which leads to $\partial udm = \partial Q + \partial W$. The volume of our shell is dV , so the change in its volume is ∂dV . $\partial W = -P\partial dV = -P\partial(1/\rho)dm$. The reason this form is more convenient is that in the end we're going to do everything per unit mass(Lagrangian form), so it is useful to have a dm instead of a dV .

The heat in a star can enter or leave the shell in two ways. They are:
1) In nuclear reaction taking place in stars it is released at the rate of q per unit mass. Thus the amount of heat in ∂t is $\partial Q_{nuc} = qdm\partial t$
2) The second way heat can enter or leave the shell is by moving down to the shell below or up to the shell above. The heat transfer can happen in three ways - radiative,convective,mechanical all of which people have heard

of earlier. But later we will discuss them in term of stars.
Let F(m) be the flux entering from beneath the shell and F(m+dm) be the flux leaving the top of shell, therefore giving us:

$$\partial Q = [qdm + F(m) - F(m + dm)]\partial t \tag{20}$$

which leads to

$$\partial Q = [qdm + F(m) - F(m) - \frac{\partial F}{\partial m}dm)]\partial t \tag{21}$$

Giving $[q - (\partial F/\partial m)]dm\partial t$. Putting in the equation earlier we get

$$\frac{du}{dt} + P\frac{d}{dt}(\frac{1}{\rho}) = q - \frac{\partial F}{\partial m} \tag{22}$$

This equation described conservation of energy for stellar material.

5.2 Energy Equation

Consider the simplest case of a star in equilibrium, so that each shell's volume and specific internal energy are constant in time. Therefore, yielding $q = \partial F/\partial m$. Integrating over all the mass, we get F(M) - F(0). We call this quantity the nuclear luminosity L_{nuc}. It is so because it has units of energy per time; i.e., it is the total rate at which nuclear reactions in the star release energy. F(0) is zero as there is no flux entering at the point where m = 0. The F(M) is the energy per unit time leaving the stellar surface making the $L_{nuc} = L$ or the total luminous energy.

We can use the equation for stars which are not exactly in equilibrium also. They are referred as time variable stars and time derivatives are not zero. Integrating them over mass:

$$\int_0^M \frac{\mathrm{du}}{\mathrm{dt}}dm + \int_0^M P\frac{\mathrm{d}}{\mathrm{dt}}(\frac{1}{\rho})dm = \int_0^M \frac{\mathrm{du}}{\mathrm{dt}}qdm - F(M) + F(0) \tag{23}$$

As m is not dependant on t we can interchange the integral and the time derivative. This gives us:

$$\int_0^M \frac{\mathrm{du}}{\mathrm{dt}}dm = \frac{d}{dt}\int_0^M udm = \frac{dU}{dt} \tag{24}$$

$$\int_0^M P\frac{\mathrm{d}}{\mathrm{dt}}(\frac{1}{\rho})dm = \int_0^M P\frac{d}{dm}(\frac{4\pi r^2 dr}{dt})dm \tag{25}$$

Integrating by - parts:

$$(P\frac{4\pi R^2 dr}{dt})_0^M - \int_0^M \frac{4\pi r^2 dr}{dt}\frac{dP}{dm}dm \tag{26}$$

We know that the first term will yield 0 as at m = 0 $dr/dt = 0$ and at m = M P = 0 approx. For the second term we can use an already derived equation:

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{dP}{dr} \tag{27}$$

Rearranging in terms of dP/dr and converting to lagrangian form dP/dm using the relation: $dP/dm = dP/dr \times (1/4\pi r^2 \rho)$ and converting the whole equation to lagrangian form we get:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} - \frac{\ddot{r}}{4\pi r^2} \tag{28}$$

Now putting this in the integral in place of dP/dm we get

$$\int_0^M \frac{Gm\dot{r}}{r^2}dm + \int_0^M \ddot{r}\dot{r}dm$$

$$\frac{Gm\dot{r}}{r^2} = -\frac{d}{dt}(\frac{Gm}{r}) \qquad \dot{r}\ddot{r} = \frac{1}{2}\frac{d\dot{r}^2}{dt}$$

which yields

$$-\frac{d}{dt}\int_0^M \frac{Gm}{r}dm + \frac{1}{2}\frac{d}{dt}\int_0^M \dot{r}^2 dm = \dot{\Omega} + \dot{T}$$

As mass is not dependant on time we can change the derivative.
As $\dot{U} + \dot{\Omega} + \dot{T} = L_{nuc} - L$
If we consider a star that is expanding or contracting extremely slowly, so that its very close to hydrostatic balance. In this case we can make two simplifications:
1) $\dot{T}$ becomes neglible compared to internal energy and the gravitational potential.
2) We can use the virial theorem for hydrostatic objects yielding $\dot{\Omega}/2 = Lnuc - L$

5.3 Nuclear Reactions

5.3.1 Chemical Evolution Equation

We have come to the point were we will discuss about what nuclear reactions take place. Stars are not composed entirely of hydrogen and helium. There is a small abundance of other heavy metals. Therefore, we will take the derivations forward by declaring the mass fraction of element i

$$X_i = \frac{\rho_i}{\rho}$$

We often want to count the number of atoms, instead of measuring the mass. Atomic mass number for species i = A_i meaning each species has a mass of $A_i m_H$. The number density of the element is $n_i = \rho_i / A_i m_H$ making mass fraction:

$$X_i = \frac{n_i A_i m_i}{\rho}$$

Some nuclear reactions also involve electrons and positrons. This have Z = 1 or Z = -1 and A = 0. If we have a reaction in the form:

$$I(A_i, Z_i) + J(A_j, Z_j) \rightleftharpoons K(A_k, Z_k) + L(A_l, Z_l) \tag{29}$$

For starters, we will acknowledge that the rate of reactions will be proportional to the rate at which the participating elements will encounter. The reaction rate per unit volume must be proportional to $n_i n_j$, where n_i and n_j are the number density of the species i and j involved in the reaction. The constant of proportionality, the reaction rate R_{ijk} where i and j element species will encounter to produce k. If the species are found to be same then the rate of reaction will not be proportional to n_i^2. Rather it will be proportional to $n_i(n_i - 1)/2$. This will be so to eliminate the repeated combinations. Therefore, the rate being approx to $n_i^2 R_{ijk}/2$.
In a reaction suppose we have number density n_i of species i which encounters species j leading to species k. As the i species is destroyed, the rate is defined as:

$$\frac{dn_i}{dt} = -n_i n_j R_{ijk}$$

There are multiple possible reactions with many possible partners, and we have to sum over all the reactions that destroy members of species i, therefore:

$$\frac{dn_i}{dt} = -\sum_{j,k} n_i n_j R_{ijk}$$

But there can also be creation of species i. If a reaction happens b/w species l and k we will have a rate of reaction:

$$\frac{dn_i}{dt} = n_l n_k R_{lki}$$

This can be alternatively written as

$$\frac{dn_i}{dt} = \frac{n_l n_k}{1+\delta_{lk}} R_{lki}$$

where,δ_{lk} is 0 if species k and l are different and 1 if same. Summing this

$$\frac{dn_i}{dt} = \sum_{l,k} \frac{n_l n_k}{1+\delta_{lk}} R_{lki}$$

Combining the rate of destruction and creation of species i we get rate:

$$\frac{dn_i}{dt} = \sum_{l,k} \frac{n_l n_k}{1+\delta_{lk}} R_{lki} - \sum_{j,k} n_i n_j R_{ijk} \tag{30}$$

The number density can be written in terms of mass fraction to obtain the equation in such a form.

5.3.2 Radiation Pressure

Let us take a look at radiation pressure.

Electromagnetic radiation exerts a minuscule pressure on everything, known as radiation pressure. In everyday situation the pressure is negligible, but in star it can become important given the vast quantities of photons emitted. Inside a star blackbody conditions exist making the radiation pressure proportional to the fourth power of temperature. As the temperatures rises internally the radiation's pressure increase thereby dominating other ones. In the most massive stars, the mass of the star is supported against gravity primarily by radiation pressure, a situation which ultimately sets the upper limit for how massive a star can become.

$$P_r = \frac{4\sigma T^4}{3c} \tag{31}$$

where, c = speed of the photons(contribution towards pressure) and σ is Stefan-Boltzman constant

6 Timescales

Stellar evolution is described by three time dependant equations, in which each deals with a different type of change and accordingly the timescale on which they change.

$$\tau = \frac{\phi}{\dot{\phi}} \tag{32}$$

Here, ϕ is a parameter which differs accordingly to the different types of timescale.

6.1 Dynamical Timescale

This describes the dynamical or structural change in the star. Therefore, the parameter we will be choosing is that of R, as that is the characteristic dimension of a spherically symmetric star. $\dot{\phi}$ is the v_{esc} (derivative of R in a gravitational field).

$$\tau_{dyn} = \frac{R}{v_{esc}} = \frac{R}{\sqrt{2GM/R}} = \sqrt{\frac{R^3}{2GM}} \tag{33}$$

Using average density $\overline{\rho}$,

$$\tau_{dyn} = \sqrt{\frac{R^3}{2GM}} \approx \frac{1}{\sqrt{G\overline{\rho}}} \tag{34}$$

Using solar scale,

$$\tau_{dyn} \approx 1000\sqrt{\frac{R^3}{R_{sun}}} \times \sqrt{\frac{M_{sun}}{M}} s \tag{35}$$

6.2 Thermal Timescale

The thermal timescale comes into play when thermal processes affect the internal energy of the star. Therefore the parameter in this case ϕ = U . Therefore, the derivative of this will be rate at which energy is radiated by the star which is $\dot{\phi}$ = L

$$\tau_{th} = \frac{U}{L} \approx \frac{GM^2}{LR} \tag{36}$$

Using solar scale,

$$\tau_{nuc} \approx 10^{15} \frac{M^2}{M_{sun}^2} \times \frac{L_{sun}}{L} \times \frac{R_{sun}}{R} s \tag{37}$$

6.3 Nuclear Timescale

The third equation deals with change in nuclear composition of the star. Now, nuclear composition goes through a lot of different processes which we will discuss later. The change in quantity by nuclear processes is a small fraction of the rest mass given by Einstein. Therefore, as $E = mc^2$ the parameter will be $\phi = \epsilon mc^2$. Again rate of change is nuclear luminosity(total rate at which nuclear reactions in the star release energy) $L_{nuc} = L$

$$\tau_{nuc} \approx \frac{\epsilon mc^2}{L_{nuc}} \approx \frac{\epsilon mc^2}{L} \tag{38}$$

Using solar scale,

$$\tau_{nuc} \approx \epsilon \times 4.5 \times 10^{20} \frac{M}{M_{sun}} \times \frac{L_{sun}}{L} s \tag{39}$$

Therefore,

$$\tau_{dyn} \ll \tau_{th} \ll \tau_{nuc} \tag{40}$$

7 Equations of stellar structure

Evolution of a star is a quasi - static process therefore the equilibrium is maintained (composition changes slowly). Let us discuss the equations of stellar equations both in Euler and Lagrangian form.

$$\frac{dP}{dr} = -\frac{\rho GM}{r^2} \qquad \frac{dP}{dm} = -\frac{GM}{4\pi r^4} \tag{41}$$

$$\frac{dm}{dr} = 4\pi r^4 \qquad \frac{dr}{dm} = \frac{1}{4\pi r^4} \tag{42}$$

$$\frac{dT}{dr} = \frac{-3}{4ac}\frac{k\rho}{T^3}\frac{F}{4\pi r^2} \qquad \frac{dT}{dr} = \frac{-3}{4ac}\frac{k}{T^3}\frac{F}{(4\pi r^2)^2} \tag{43}$$

$$\frac{dF}{dr} = 4\pi r^2 \rho q \qquad \frac{dF}{dm} = q \tag{44}$$

First represents hydrostatic equilibrium equation.
Second represents continuity equation.
Third represents radiative transfer equation.
Fourth represents thermal equilibrium equation.

7.1 Polytropes

It turns out that many of the observed properties of stars, except their lifetimes and radii, reflect chiefly the need to be in hydrostatic and thermal equilibrium, and not the energy source.
Polytropes is a concept used by physicists to understand the stellar structure of the stars. Polytropes are self-gravitating gaseous spheres in which the pressure depends on density in the form of $P = K\rho^{(n+1)/n}$. The n is known as polytropic index. Basically it is a solution of Lane - Emden equation which we will derive right now.

Let us begin with HSE equation
$dP/dr = -\rho GM/r^2$
$dP/dr \times r^2/\rho = -GM$

Differentiating agin wrt r

$$\frac{d}{dr}[\frac{r^2}{\rho}\frac{dP}{dr}] = -G\frac{dM}{dr} = -4\pi G r^2 \rho \tag{45}$$

$$\frac{1}{r^2}\frac{d}{dr}[\frac{r^2}{\rho}\frac{dP}{dr}] = -4\pi G\rho \tag{46}$$

The Lane - Emden equation is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid(in this case of a star). So let us make this dimensionless. Central density is ρ_c and then $\rho = \rho_c\theta^n$.

$P = k(\rho_c\theta^n)^\lambda$

$$\frac{1}{r^2}\frac{d}{dr}[\frac{r^2}{\rho}\frac{dP}{dr}] = -4\pi G\rho \tag{47}$$

$$\frac{1}{r^2}\frac{d}{dr}[\frac{r^2}{\rho_c(\theta)^n}\frac{dk(\rho_c)^\lambda(\theta)^{n\lambda}}{dr}] = -4\pi G\rho_c(\theta)^n \tag{48}$$

Taking the constants out and replacing the λ with (n+1)/n we get

$$[\frac{P_c(n+1)}{4\pi G{\rho_c}^2}]\frac{1}{r^2}\frac{d}{dr}[\frac{r^2 d\theta}{dr}] = -\theta^n \tag{49}$$

As the quantity in brackets has a dimension of $length^2$, therefore we assign a value α^2 to the quantity in the brackets. We will also assign $r = \alpha\xi$ which leads to

$$\frac{1}{\xi^2}\frac{d}{d\xi}[\xi_2\frac{d\Theta}{d\xi}] = -\Theta^n \tag{50}$$

And tada we got the Lane - Emden equation. The two central boundary conditions are now $\Theta = 1$ and $d\Theta/d\xi = 0$ at $\xi = 0$

Putting n = 0 in the equation (43) we get $(d/d\xi)(\xi^2 d\Theta/d\xi) = -\xi^2$
Integrating wrt ξ we get

$$\Theta = -\frac{\xi^2}{6} - \frac{C}{\xi} + D \tag{51}$$

Applying the boundary conditions that $\Theta = 1$ and $d\Theta/d\xi = 0$ at $\xi = 0$, we immediately see that we must choose $C = 0$ and $D = 1$; so the solution is therefore $\Theta = 1 - (\xi^2/6)$
It is obvious the function is monotonically decreasing when $\xi > 0$ and becomes 0 at $\xi = \xi_1 = \sqrt{6}$

The radius at which ξ reaches zero is r = R, therefore $R = \alpha\xi$

Computing the mass of the star we get,

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\rho_c \int_0^{\xi_1} \xi^2\Theta^n dr = -4\pi\rho_c \int_0^{\xi_1} \frac{d}{d\xi}[\xi^2\frac{d\Theta}{d\xi}] \tag{52}$$

We get

$$M = -4\pi\alpha^3\rho_c \times \xi_1^2[\frac{d\Theta}{d\xi}] \tag{53}$$

We also use this to calculate how much dense is the star centrally to mean density. $D_n = \rho_c/\overline{\rho} = \rho_c 4\pi R^3/3M$ Use the already mentioned derivations we get

$$D_n = -\frac{3d\Theta}{\xi_1 d\xi}^{-1} at\xi = \xi_1 \tag{54}$$

A second useful relationship is between mass and radius. We start by expressing the central density $_c$ in terms of the other constants and length scale α.

$$\rho_c = \frac{(n+1)K^{n/n-1}}{4\pi G\alpha^2} \tag{55}$$

Next we substitute this into the equation for the mass:

$$M = -4\pi\alpha^3 \times \xi_1^2 \frac{d\Theta}{d\xi} \times \frac{(n+1)K}{4\pi G^2}^{n/n-1} \tag{56}$$

This is at $\xi = \xi_1$ and doing substitution we relate mass and radius.
$M \approx R^{(n-3)/(n-1)}$

A third useful expression is for the central pressure. From the equation of state : $P_c = K\rho_c^{n+1/n}$

$$P_c = \frac{(4\pi G)^{1/n}}{n+1} \times \frac{GM}{-\xi^2(d\Theta/d\xi)}^{(n-1)/n} \times \frac{R}{\xi_1}^{3-n} \times \rho_c^{(n+1)/n} \tag{57}$$

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \tag{58}$$

8 Physics of stellar fluids

In this section we will discuss deeply about different forms of pressure and how they affect the total pressure and the physics related to each pressure. The physics of stellar interior deals with :

Properties of gaseous systems
Radiation and it's effects
Interaction b/w gas and radiation

The equation of state is a relation b/w the exerted pressure, present temperature and density. For understanding the stars, let us assume that the gas is ideal as we have done before and here we will prove how the assumption is correct. For a gas to be ideal the particles have to be non-interactive and obeying the gas laws exactly. At the prevailing temperatures it is expected that coulomb interactions will take place. But here we will see that the kinetic energy of the particles is large when compared with coulombic forces.

$d = (Am_H/\overline{\rho})^{1/3} = (4\pi Am_H/3M)^{1/3}R$

Using the d(mean distance) in equation $E_c = ((Ze)(e)/4\pi\epsilon d)$ we get $E_c/k_B\overline{T} << 1$ and substituting $\overline{T}$ with $\overline{T} = (\alpha/3) \times (\mu/R) \times (GM/R)$ (which we took care of while discussing virial theorem) and replace μ with A

Therefore,

$$\frac{E_c}{k_B\overline{T}} = \frac{1}{4\pi\epsilon}\frac{Z^2e^2}{G(Am_H)^{4/3}M^{2/3}} \approx 0.011\frac{Z^2}{A^{4/3}}\frac{M}{M_{sun}}^{-2/3} \tag{59}$$

An important point ,i.e., for Z = 1, A = 1 and for Z higher A = 2Z the ratio is well below 1 but the problem arises when M $\approx$M$_{sun}$ $\times$ 10^{-3} or lower the ratio is approx 1. Now the stars may not be of this mass but planets are like Jupiter. We can conclude from here that stars are composed of gases that show ideal features and as the planet's mass reduces the closer it is to being of a solid structure.

8.1 Kinetic Theory Model of Pressure

The pressure is the force exerted by a gas on the surface. Here we will use for pressure - momentum per unit time per unit area. The reason there is a momentum transfer is that particles in the gas are moving around at random, and that some of them will strike the walls of the vessel, bounce off, and transfer momentum. We can compute the pressure by computing this momentum transfer.

When a particle with an angle θ, momentum p encounters and bounces elastically off an immobile surface. The momentum transferred is $2pcos\theta$. Talking in collective sense, a beam of particles will encounter the surface with momentum p, number density n and velocity ν. It will give a surface strike rate at area dA - $n\nu cos\theta dA$. Total rate at which the beam transfers the momentum is :

$$\frac{d^2p_{surf}}{dtdA} = 2n\nu pcos^2\theta \tag{60}$$

Now, as the beam will travel in all direction we will take a strip in θ and $\theta + d\theta$ relative to the normal, where the number density will be denoted as $d(n)\theta/d\theta$. The solid angle of the strip taken will be $2\pi sin\theta d\theta$ and 4π for the total.
$d(n)\theta/d\theta = (1/2)nsin\theta$
Therefore the momentum transferred =

$$\frac{d^2p_{surf}}{dtdA} = np\nu\int_0^{\pi/2} cos^2\theta sin\theta d\theta = \frac{n\nu p}{3} \tag{61}$$

We let $dn(p)/dp$ be the number of particles with momenta between p and $p + dp$. The total pressure will be:

$$P = \int_0^\infty \frac{1}{3}\frac{d(n)p}{dp} p\nu dp \tag{62}$$

8.2 Types of Pressure

1) Ideal gas law
The star is composed of gases which are ionized. The ideal gas law therefore helps in determining the pressure. Let a gas has mass of all particles m and have a Maxwell-Boltzman velocity distribution. Here, we know that the probability of the gas having energy E is proportional to e_{-E/k_BT}. Taking the route of three dimensionality for the momentum calculation of stellar gases.

$$E = \frac{p^2}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

The probability for the momentum to lie b/w p and p+dp in a volume of $4\pi p^2 dp$, giving us

$$\frac{dn(p)}{dp} \propto 4\pi p^2 e^{-p^{2/2mk_BT}}$$

Integrating it we get:

$$n = 4\pi k \int_0^{infty} p^2 e^{-p^{2/2mk_BT}} dp$$

Keeping q $= p/\sqrt{2mk_BT}$ we integrate to get

$$n = k(2\pi m k_B T)^{3/2}$$

In this integration one can choose the way of multivariable calculus. k here is constant of proportionality. Now we can use k in $d(n)p/dp$ giving us

$$\frac{d(n)p}{dp} = \frac{4\pi n}{(2\pi k_B mT)^{3/2}} p^2 e^{-p^2/2mk_BT} \tag{63}$$

This can be used to calculate the pressure:

$$P = \int_0^{infty} \frac{1}{3} \frac{4\pi n}{(2\pi k_B mT)^{3/2}} p^2 e^{-p^2/2mk_BT} p \frac{p}{m} dp$$

Taking the constants out we will observe:

$$P \propto \int_0^\infty q^4 e^{-q^2} dq$$

q represents the same quantity earlier considered and same integration method will be used. Therefore, giving us $P = nk_BT$

2) Multiple species in gases

Stellar gases are composed of more than one species. Each species follows the Boltzmann distribution, and the sum of the momentum transferred to a surface is simply the sum of the momenta transferred by the particles of each species, each of which is given by nk_BT. Therefore, for n species the total pressure is

$$P = (\sum_{i=1}^{N} n_i)k_BT$$

Using mass fraction of i species in n_i we get:

$$P = \sum_{i=1}^{N} \frac{X_i}{A_i}\rho RT =$$

$k_B/m_H = R$

$$\sum_{i=1}^{N} \frac{X_i}{A_i} = \frac{1}{\mu}$$

μ will depend upon the composition of the gas and the state of ionization in the star.
neutral $\mu = 1$ fully ionized $\mu = 0.5$
For stars we will have to consider both neutrons and electrons for the density calculation.

$$Hydrogen - Neutron = \frac{X\rho}{m_H} \qquad electron = \frac{X\rho}{m_H} \tag{64}$$

$$Helium - Neutron = \frac{Y\rho}{4m_H} \qquad electron = \frac{2Y\rho}{4m_H} \tag{65}$$

$$Others - Neutron = \frac{Z\rho}{Am_H} \qquad electron = \frac{A}{2}.\frac{Z\rho}{Am_H} \tag{66}$$

Where,
X hydrogen - mass m_H, one electron
Y helium - mass $4m_H$, two electrons
Z the rest, 'metals', average mass Am_H, approximately (A / 2) electrons per nucleus
Note, that we refer to metals as elements other than H and He in astronomy usually - so even C would be considered as one. Assuming A>>1.

First we will calculate pressure due to ions

$$\frac{1}{\mu_I} = \frac{X}{1} + \frac{Y}{4} + \frac{Z}{A_{metals}}$$

For pressure due to electrons we will proceed further. there is one free electron per proton. If n_i is the number density of ions of species i, then the number density of electrons :

$$n_e = \sum_1 Z_i n_i = \sum_1 Z_i \frac{X_i}{A_i}$$

where

$$\frac{1}{\mu_e} = \sum_1 Z_i \frac{X_i}{A_i}$$

$$X + \frac{Y}{2} + \frac{Z}{Z}A = \frac{1}{\mu_e}$$

$$P = P_I + P_e \qquad \frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}$$

3) Relativistic gases

The kinetic theory here is needed to generalize the concept of pressure to gasses that are not ideal, classical gasses. The particles in such has a velocity nearly equally to c. Here energy will be defined as $E = pc$. For electrons, the transition between the two regimes occurs when $(3/2)k_BT$ be- comes comparable to $(1/2)m_ec2$, the electron rest energy. Therefore relativistic conditions occur at temperatures of:

$$\frac{3k_BT}{2} = \frac{m_ec^2}{2} \qquad T_{rel} \approx \frac{m_ec^2}{3k_B} = 2 \times 10^9$$

The momentum distribution is described by the equation:

$$\frac{d(n)p}{dp} = 4\pi kp^2 e^{-E/k_BT} = 4\pi kp^2 e^{-pc/k_BT}$$

As done earlier we will calculate k using integration

$$n = 4\pi k \int_0^\infty p^2 e^{-pc/k_BT} dp$$

The integration is simple and will yield

$$n = 8\pi k \frac{(k_BT)^3}{c^3}$$

which gives

$$k = \frac{c^3}{(k_BT)^3} \frac{n}{8\pi}$$

9 Internal Energy

The distribution of particle momenta, $dn(p)/dp$, which was calculated from the Boltzmann distribution is also important for internal energy(particles being non-interactive, the K.E. of moving particles is the internal energy).

Density of energy within that volume of space:

$$e = \int_0^\infty \frac{dn(p)}{dp} \epsilon(p) dp$$

$\epsilon(p)$ being the energy of a particle with momentum p. Now dividing energy per unit mass is e/ρ.

$$u = \frac{1}{\rho} \int_0^\infty \frac{dn(p)}{dp} \epsilon(p) dp$$

$$\epsilon(p) = mc^2(\sqrt{1 + \frac{p^2}{(mc)^2}} - 1)$$

In these cases we use two cases :

1) Non-relativistic(p¡¡ mc) and $\epsilon(p) = p^2/2(mc)^2$ (Taylor expansion used)
2) Ultra-relativistic(p¿¿mc) and $\epsilon(p) = pc$

We will have four pairs of condition pertaining to - relativistic, non-relativistic, degenerate, non-degenerate

Relativistic, non-degenerate limits

$$u = \frac{1}{\rho}\int_0^\infty (\frac{c}{k_BT})^3 \frac{n}{2} p^2 e^{-pc/k_BT}(pc)dp$$

$$= \frac{c^4}{2mk_BT}\int_0^\infty p^3 e^{-pc/k_BT}dp = \frac{3}{mk_BT}$$

$$u = \frac{3P}{\rho} \tag{67}$$

Non-relativistic, non-degenerate limits

$$u = \frac{1}{\rho}\int_0^\infty \frac{4\pi n}{(2\pi mk_BT)^{3/2}} p^2 e^{-p^2/2\pi mk_BT}\frac{p^2}{2m}dp$$

$$= \frac{2\pi}{m^2(2\pi mk_BT)^{3/2}}\int_0^\infty p^4 e^{-p^2/2\pi mk_BT}dp$$

$$= \frac{4k_BT}{m\pi^{1/2}}\int_0^\infty q^4 e^{-q^2}dq$$

$$u = \frac{3k_BT}{2m} = \frac{3}{2}\frac{P}{\rho} \tag{68}$$

Non-relativistic, degenerate limits

$$u = \frac{1}{\rho}\int_0^{\rho_o} \frac{8\pi p^2}{h^3}(\frac{p^2}{2m})dp$$

$$= \frac{4\pi}{5m\pi h^3}(\rho_o^5) = \frac{(3^{2/3})}{(\pi^{2/3})}\frac{3h^2 n^{5/3}}{40\rho m}$$

$$u = \frac{3}{2}\frac{P}{\rho} \tag{69}$$

Relativistic, degenerate limits

$$u = \frac{1}{\rho}\int_0^{\rho_o} \frac{8\pi p^2 (pc)}{h^3} dp$$

$$= \frac{3^{1/3}}{\pi^{1/3}}\frac{3}{8}\frac{hcn^{4/3}}{\rho}$$

$$u = 3\frac{P}{\rho} \tag{70}$$

9.1 Adiabatic Process

Adiabatic processes take place in a star. Any process which takes place on a timescale shorter than Kelvin-Helmholtz scale is thought of as adiabtic. Let us discuss the reason. Any gas which cannot exchange heat with the environment or extract it from internal sources(in case of stars it is nuclear burning). But still the energy exchange can take place through radiation. The reason we do not count it in is due to the long time it takes as opposed to dynamical time(change in composition). Therefore, we consider gas in the star to be adiabatic. Onto the thermodynamic eq:

$$\frac{du}{dt} + P\frac{d}{dt}(\frac{1}{\rho}) = q - \frac{\partial F}{\partial m}$$

As heat exchange becomes zero

$$\frac{du}{dt} + P\frac{d}{dt}(\frac{1}{\rho}) = 0$$

For solving the differential equation consider $u = \phi P/\rho$, where ϕ is a constant that depends on the type of gas. Substituting and solving:

$$\frac{d}{dt}(\frac{\phi P}{\rho}) + P\frac{d}{dt}(\frac{1}{\rho}) = 0$$

$$\frac{d}{dt}(\frac{\phi P}{\rho}) + P\frac{d}{dt}(\frac{1}{\rho}) = 0$$

$$(\frac{\phi}{\rho})\frac{dP}{dt} + \phi P\frac{d}{dt}(\frac{1}{\rho}) + P\frac{d}{dt}(\frac{1}{\rho}) = 0$$

Grouping terms

$$(\frac{\phi}{\rho})\frac{dP}{dt} + P(\phi+1)\frac{d}{dt}(\frac{1}{\rho}) = 0$$

$$(\frac{\phi}{\rho})\frac{dP}{dt} = -P(\phi+1)\frac{d}{dt}(\frac{1}{\rho})$$

$$(\frac{\phi}{\rho})\frac{dP}{dt} = \frac{P(\phi+1)}{\rho^2}\frac{d\rho}{dt}$$

$$\frac{dP}{P} = \frac{(\phi+1)}{\phi}\frac{d\rho}{\rho}$$

$$\ln P = \gamma_a \ln\rho + \ln K_a \tag{71}$$

$$P = \rho^{\gamma_a} K_a \tag{72}$$

$\gamma_a = (\phi+1)/\phi$ and K_a is a constant
The constant of integration K_a is called the adiabatic constant, and it is determined by the entropy of the gas.

Our focus in this section has been on adiabatic process and the index. The index plays an important role in stars. The index gives an idea of how much the gas is resistant to compression and plays an important role in the structure of the stars. All stars have $\gamma_a = 5/3$(close) in a non-relativistic star but as it approaches $\gamma_a = 4/3$ the star becomes relativistic and resistance to compression drops.

Radiative effects cause interstellar gas clouds to act as if they had $\gamma_a = 1$ making the gas weakly resistant to compression therefore leading to the stage of protostars(early stage of star formation).

9.2 Adiabatic index for partially ionized gas

The stellar gases are not so simple. They are complex but the index can be generalized.

This case is tricky because the number of free gas particles itself becomes a function of temperature, and because the potential energy associated with ionization and recombination becomes an extra energy source or sink for the gas. Consider a gas of pure hydrogen within which the number density of neutral atoms is n0 and the number densities of free protons and electrons are $n_p = n_e$. The number density of all atoms regardless of their ionization state is $n = n_e + n_o$. The ionization fraction is:

$$x = \frac{n_e}{n}$$

The pressure on gas is dependant on the free particles:

$$P = n_e k_B T + n_p k_B T + n_o k_B T = (n_o + n_p)(1+x)RT \tag{73}$$

Thus the pressure at fixed temperature is higher if the gas is more ionized, because there are more free particles. Let us see Saha equation. According to it:

$$\frac{n + n_e}{n_o} = \frac{g}{h^3}(2\pi m k_B T)^{3/2} e^{-XkT} \tag{74}$$

Here X is the ionization potential

$$\frac{x^2}{1-x^2} = \frac{g}{h^3}(2\pi m k_B T)^{3/2} e^{-XkT}$$

Therefore,

$$u = \frac{3}{2}\frac{P}{\rho} + \frac{x}{X} m_H \tag{75}$$

Differentiating the equation and using in the first thermodynamic equation

$$\frac{3}{2}(\frac{1}{\rho})dP + \frac{3}{2}Pd\frac{1}{\rho} + \frac{X}{m_H}\frac{\partial x}{\partial \rho}d(\rho) + \frac{X}{m_H}\frac{\partial x}{\partial P}d(P) + Pd(\frac{1}{\rho}) = 0$$

Rearranging the terms we will get:

$$(\frac{3}{2} + \frac{X}{kT}\frac{P}{1+x}\frac{\partial x}{\partial P})\frac{dP}{P} = (\frac{5}{2} - \frac{X}{kT}\frac{\rho}{1+x}\frac{\partial x}{\partial \rho})\frac{d\rho}{\rho} \tag{76}$$

Integrating the terms we will get:

$$\gamma_a(x) = \frac{5 + (\frac{5}{2} + \frac{X}{kT})^2 x(1-x)}{3 + (\frac{3}{2} + \frac{X}{kT})^2 x(1-x)} \tag{77}$$

We have come far from where we started. Just a little topic and then we will be off to the real magic.

10 Nuclear Reactions in Stars

The nuclear process happening in the stars is the actual energy generator. Whenever we talk about nuclear processes we have to talk about the binding energy. It is known that the mass is not conserved in nuclear reactions - the difference depending on the binding energy of the interacting elements. Let us take a nuclear reaction:

$$I(A_i, Z_i) + J(A_j, Z_j) \rightleftharpoons K(A_k, Z_k) + L(A_l, Z_l)$$

Now, the energy Q released will be

$$Q = (M_i + M_j - M_k - M_l)c^2$$

Taking it in a different form:
$Q = ((M_i - A_i m_H) + (M_j - A_j m_H) - (M_k - A_k m_H) - (M_l - A_l m_H))c^2 + (A_i m_H + A_j m_H - A_k m_H - A_l m_H)c^2$
Due to law of conservation of baryons in a reaction(baryons here deal with protons and neutrons) the 2nd part of right hand equation will become zero. Therefore yielding, $\Delta M(I) = (M_i - A_i m_H)c^2$, the change is known as mass excess(positive or negative).
Refreshing our memories, rate per unit volume at which reaction takes place

$$\frac{\rho^2}{m_H^2} \frac{1}{1+\sigma_{ij}} \frac{X_i}{A_i} \frac{X_j}{A_j} R_{ijk}$$

and the total nuclear energy release rate per unit volume is

$$\frac{\rho^2}{m_H^2} \sum_{ijk} \frac{1}{1+\sigma_{ij}} \frac{X_i}{A_i} \frac{X_j}{A_j} R_{ijk} Q_{ijk}$$

10.1 Rate of reaction

For nuclear reactions to take place we need the microphysical details to tick some basic boxes. We know about the coulomb barrier, will read about quantum tunneling and discuss about a few more missing blanks.

10.1.1 Coulomb Barrier

The stars are a powerhouse undergoing nuclear fusion. When we talk about fusion reactions we cannot ignore two main points that the reactions are governed by:
1) Long-range repulsive electrostatic Coulomb force

2) Short-range attractive strong nuclear force

For the fusion to take place the interacting nuclei should overcome the coulomb barrier(due to positive charges) that arises due to repulsion. Let us take an example of nucleus with charge Z_i and Z_j. Now,the potential energy will be:

$$U = \frac{Z_i Z_j e^2}{r}$$

11 Forms of nuclear reaction